

A projection formula for the ind-Grassmannian

Erik Carlsson

March 25, 2013

Abstract

Let $X = \bigcup_i X_i$ be the ind-Grassmannian of codimension n subspaces of an infinite-dimensional torus representation. If \mathcal{Y} is a bundle on X , then $\sum_j (-1)^j \Lambda^j(\mathcal{Y}^*)$ morally represents the K -theoretic fundamental class $[\mathcal{O}_Y]$ of a subvariety $Y \subset X$, though such a class may not be defined. It is desirable to lift a K -theoretic “projection formula” from the finite-dimensional subvarieties X_i , but this is obstructed by the ambiguity in the order of the limits in i and j . We find conditions in which the projection formula does in fact hold, and consider examples in which Y is the Hilbert scheme of points in the plane, the moduli space of higher rank sheaves on \mathbb{P}^2 (instantons), the Hilbert scheme of an irreducible curve singularity, and the affine Grassmannian of $SL(2, \mathbb{C})$. In the last example, the projection formula becomes an instance of the Weyl-Kač character formula, which has long been recognized as the result of formally extending Borel-Weil theory and localization to Y [47].

1 Introduction

Let X be a smooth complex projective variety, let

$$T \curvearrowright X, \quad T = (\mathbb{C}^*)^d = \{(z_1, \dots, z_d)\},$$

be a complex torus action on X , and let \mathcal{E} be an equivariant bundle on X . The K -theoretic Atiyah-Bott-Lefschetz localization formula describes the character of the derived push forward to a point, also known as the equivariant Euler characteristic,

$$\chi_X(\mathcal{E}) = \sum_i (-1)^i \operatorname{ch} H_X^i(\mathcal{E}) = \sum_{C \subset X^T} \chi_C \left(\mathcal{E}_C \lambda(\mathcal{N}_{X/C}^*)^{-1} \right), \quad (1)$$

Here H^i is the Čech cohomology group, ch is the (Chern) character map, C ranges over the fixed components of the torus action, $\mathcal{N}_{X/C}$ is its normal bundle, and λ is the usual operation on $K(C) \otimes \mathbb{C}[T]$ defined below (4). The restriction \mathcal{E}_C may be interpreted as the restriction to C , combined with the evaluation at a generic point $z \in T$, so that equation (1) is an equality of functions on T . See [12] for a reference.

Suppose $Y \subset X$ is an invariant subvariety which is the zero set of an equivariant section of a bundle \mathcal{Y} . Then the fundamental class $[\mathcal{O}_Y]$ is given by $\lambda(\mathcal{Y}^*) \in K_T(X)$, and we have the projection formula

$$\chi_Y(\mathcal{E}_Y) = \chi_X(\mathcal{E}\lambda(\mathcal{Y}^*)). \quad (2)$$

If we apply the localization formula to either side, the resulting identity is not mysterious; The fundamental class $\lambda(\mathcal{Y}^*)$ vanishes when restricted to a component $C \subset X$ that does not intersect with Y , and the two expressions are in fact equal termwise. However, if a more tractable formula for χ_X is known, then equation (2) produces a tractable formula for χ_Y evaluated on pullbacks from X . Tractable might mean a formula that is presented as a Laurent polynomial or power series with integer coefficients, as opposed to an unworkable rational function coming from localization.

We will show that the projection formula may be extended to many examples in which Y is an interesting moduli space, and X is the Grassmannian of codimension n subspaces of an infinite-dimensional torus representation Z , defined as an ind-variety

$$\cdots \subset X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X, \quad \bigcup_i X_i = X,$$

where each X_i is finite-dimensional. We include the following examples:

1. Y is the Hilbert scheme of n points in the complex plane, and $X = G(n, R)$, the Grassmannian of codimension n subspaces of the total space of the ring $R = \mathbb{C}[x, y]$. The imbedding is the map which associates to a subscheme of \mathbb{C}^2 the total space of the corresponding ideal. The projection formula becomes a power series expansion with integer coefficients for the Euler characteristic of a subbundle

$$\mathcal{E} \subset \mathcal{U} \otimes \cdots \otimes \mathcal{U},$$

where \mathcal{U} is the tautological n -dimensional bundle on Y .

2. $Y = \mathcal{M}_{r,n}$ is the moduli space of higher rank sheaves (instantons) on \mathbb{P}^2 framed at infinity, and X is a Grassmannian of codimension n subspaces of a free module of rank r over R . There is no actual imbedding here, but we still find a class in K -theory $\lambda(\mathcal{Y}^*)$ such that the right hand side of (2) agrees with the desired localization on Y .
3. Y is the Hilbert scheme of a plane curve singularity $y^2 = x^3$, and X is the Grassmannian of codimension n subspaces of the ring

$$\mathbb{C}[x, y]/(y^2 - x^3) \cong \mathbb{C}[u^2, u^3].$$

The projection formula provides an analogous expression to example 1.

4. Y is the affine Grassmannian of the loop group of $SL(2, \mathbb{C})$, and X is the Sato Grassmannian of half infinite-dimensional subspaces of a faithful representation of LG . In this case, the projection formula produces an instance of the Weyl-Kač character formula, which is predicted by formally extending Borel-Weil theory and localization to loop groups. This idea was discussed by Segal in [47], and has been studied by several authors, including generalizations to the analogous flag varieties [26, 49, 50].

In the last two examples we have confined ourselves to special cases for simplicity, but they can obviously be generalized to some extent.

We must define what is meant by χ_Y , because we are interested in instances in which Y is infinite-dimensional, noncompact, or singular, as we have seen. In this paper, Y is only defined implicitly through the class $\lambda(\mathcal{Y}^*)$, for some virtual bundle \mathcal{Y}^* . Its Euler characteristic is defined formally by (2),

$$\chi_Y(\mathcal{E}_Y) = \sum_{C \subset X^T} \chi_C \left(\mathcal{E}_C \lambda(\mathcal{Y}_C^*) \lambda(\mathcal{N}_{X/C}^*)^{-1} \right),$$

even when there is no actual variety Y in mind. For this expression to be well-defined, we must first know that $\lambda(\mathcal{Y}_C^*)$ exists, and vanishes for all but finitely many components C .

We decide that $\lambda(\mathcal{Y}^*)$ represents an actual variety Y , if we have agreement between the two definitions of χ_Y . This is manifested combinatorially in the manner described in the second paragraph; the class $\lambda(\mathcal{Y}^*)$ combines with $\lambda(\mathcal{N}_{X/C}^*)$ to produce the normal bundle $\mathcal{N}_{Y/C}^*$ on all fixed components that intersect Y , and vanishes on all

others. For instance, in example 1, we consider a standard torus action on Z , such that an invariant subspace $V \in G(n, Z)$ is represented by a subset $S \subset \mathbb{N}^2$ of size n , corresponding to the torus weights of its complement. We will see by direct calculation that $\lambda(\mathcal{Y}^*)$ vanishes at all points unless S happens to be the contents of a Young diagram, so that V is the total space of an ideal $I \subset R$, representing torus-fixed point in the Hilbert scheme.

The obstacle to lifting the projection formula turns out to be a matter of switching two limits. The class $\lambda(\mathcal{Y}^*)$ is not a well defined element of $K_T(X)$ defined as an inverse limit, but $\lambda^i(\mathcal{Y}^*)$ is always defined for any i , and one might approximate $\chi_X(\mathcal{E}\lambda(\mathcal{Y}^*))$ by

$$\chi_{ij}(z) = \sum_{k \leq j} (-1)^k \chi_{X_i}(\mathcal{E}\lambda^k(\mathcal{Y}^*)), \quad z \in T.$$

We can see that

$$\begin{aligned} \chi_Y(\mathcal{E}_Y) &= \lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} \chi_{ij}(z), \\ \sum_j (-1)^j \chi_X(\lambda^j(\mathcal{Y}^*)) &= \lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} \chi_{ij}(z). \end{aligned}$$

However, it is not clear that the limits are switchable. In general, by Serre duality, the characters of the higher cohomology groups should contribute a very large negative power of z which should dominate the other terms for $j \gg i$, but vanish for $i \gg j$. As a result, the two expressions should almost never be equal.

We find conditions for when the projection formula holds when \mathcal{Y}^* is the virtual bundle obtained by tensoring the cotangent bundle on X by torus characters

$$\mathcal{Y}^* = (1 - M)T_X^*, \quad M \in K(T).$$

In the examples above, Z can be seen to be the total space of an equivariant module over a polynomial algebra R . The desired bundle \mathcal{Y}^* , is obtained choosing M to be the polynomial such that the character of R is M^{-1} .

Theorem A. *Suppose $T = \{z\}$ is a one-dimensional complex torus acting on an infinite-dimensional vector space Z , and let X_i be the components of the ind-Grassmannian of codimension n subspaces of Z . If the conditions described in theorem 1 below are satisfied, then*

$$\chi_Y(\mathcal{E}_Y) = \sum_{j \geq 0} (-1)^j \chi_X(\mathcal{E}\lambda^j(\mathcal{Y}^*)).$$

The most interesting condition is condition 2, which imposes bounds on the dimensions of the weight spaces of Z . These conditions tend to be satisfied when Z is the total space of an equivariant module over a ring with character M^{-1} .

The motivations for this paper have to do with a fascinating and well-studied interplay between the Hilbert scheme of points on a surface, representation theory, and modular forms. In many different studies, geometric correspondences between the Hilbert schemes of different points induce an action of various infinite-dimensional Lie algebras on \mathcal{H} , the direct sum of the cohomology groups of $\text{Hilb}_n S$ over all n , see [4, 10, 17, 30, 28, 35] to name a few. There is a related story in K -theory which in many cases is based on Haiman's character theory of the Bridgeland King and Reid isomorphism which identifies $K(\text{Hilb}_n \mathbb{C}^2)$ as an inner-product space with the ring of symmetric polynomials in infinitely many variables [6, 11, 41, 46]. In some cases, the resulting character theory leads to functional properties of the generating function of cohomological or K -theoretic constants in a variable q , over the number of points n [8, 10, 25, 53].

These phenomena are closely related to a physical conjecture known as AGT (Alday, Gaiotto, Tachikawa) [1], which connects correlation functions in four dimensional gauge theory with a certain Liouville theory. In fact, there are two current mathematical proofs of this conjecture that proceed along these lines [34, 52]. It would be very desirable mathematically and physically to discover integrals on a larger moduli space which restrict to both sides of this dictionary under different specializations of the equivariant parameters. The motivation in extending the projection formula is that interesting integrands on a Grassmannian manifold are simply easier to construct than interesting moduli spaces. A second issue is that Haiman's theory makes sense when the moduli space is the Hilbert scheme, whereas the structures on the cohomology and K -theory of $\mathcal{M}_{r,n}$ also lead to interesting character theory. A fundamental example is the action of the Kaç-Moody algebra $\widehat{sl_r \mathbb{C}}$ on $H^*(\mathcal{M}_{r,n})$ [30, 40, 41, 42], which prompted the Kaç-Moody example.

Acknowledgements. The author would like to thank the Simons foundation, for its support, as well as Hiraku Nakajima, Alexei Oblomkov, and Andrei Okounkov, for many valuable discussions.

2 Contour integrals and the Grassmannian

Let $K_T(X)$ denote the complex equivariant K -theory of a smooth complex projective variety with an action of $T = (\mathbb{C}^*)^d$, and let λ^i denote the usual operation defined on bundles by

$$\lambda^j([\mathcal{E}]) = [\Lambda^j(\mathcal{E})].$$

The total operation is defined by

$$\lambda(\gamma) = \lim_{w \rightarrow 1} \lambda^j(w\gamma) = \lim_{w \rightarrow 1} \sum_j (-1)^j w^j \lambda^j(\gamma) \quad (3)$$

where the limit is the analytic continuation to $w = 1$ of the rational function defined by the right hand side for w near zero. The limit exists if $\gamma = [\mathcal{E} - \mathcal{F}]$ for honest bundles \mathcal{E}, \mathcal{F} , with $\lambda(\mathcal{F})$ invertible in $K(C)$, and equals $\lambda(\mathcal{E})\lambda^{-1}(\mathcal{F})$.

If $f = \sum_I a_I x^I$ for $a_I \in \mathbb{Z}$, and x^I are monomials in some set of indeterminants, we have the usual definition

$$\lambda(f) = \prod_I (1 - x^I)^{a_I}. \quad (4)$$

If $z = \{z_i\}$ is some set of variables, we define

$$\lambda(z) = \lambda\left(\sum_i z_i\right) = \prod_i (1 - z_i).$$

In this paper, every variable will be considered plethystic, meaning it counts as an indeterminant for the purposes of (4). This includes variables such as z, w below which are expected to have fixed complex values.

We may also apply this to an element of $\mathbb{C}((z))$, such as

$$\lambda\left(\frac{z}{1-z}\right) = \prod_{i \geq 1} (1 - z^i)^{-1},$$

for a given value $|z| < 1$. This is relevant to the main equation (10), in which case the conormal bundle $\mathcal{N}_{X/C}^*$ represents an element of $\mathbb{C}((z)) \otimes K(C)$ for a fixed component C of the infinite Grassmannian X . We will regard such a class as a function on a neighborhood of the

origin in \mathbb{C} with values in $K(C)$, and consider the image at a chosen value $z \in T$. See [12] for a reference.

Suppose Z is a finite-dimensional representation of $T = (\mathbb{C}^*)^d$, and consider the Grassmannian variety of codimension n subspaces of Z ,

$$X = G(n, Z) = \{V \subset Z \mid \text{codim}(V) = n\}.$$

There is a tautological bundle \mathcal{V} whose fiber over $V \subset Z$ is V itself, and a rank n quotient bundle $\mathcal{U} = \mathcal{Z}/\mathcal{V}$, where $\mathcal{Z} = G(n, Z) \times Z$. The action of T on Z induces an action on the Grassmannian, and on the above bundles.

If ρ is a finite-dimensional representation of GL_n , then $\mathcal{E} = \rho(\mathcal{U})$ is a new bundle with fiber $\rho(U)$. More generally, if $f \in K(GL_n)$, then $f(\mathcal{U})$ defines an element of $K_T(X)$. Any element $f \in \Lambda_n$, the ring of symmetric functions in n variables, defines an element of $K_{pol}(GL_n)$, the K group of representation of GL_n with polynomial matrix elements. In this case $f(\mathcal{U})$ is in the span of subbundles of $\mathcal{U}^{\otimes k}$ for $k = \deg(f)$.

Suppose $n = 1$ and fix a point $(z_1, \dots, z_d) \in T$, so that the Euler characteristic and the weights of Z are just numbers. Then the Euler characteristic may be neatly described by contour integrals

$$\chi_X(\mathcal{E}) = \left(\int_{|x|=\epsilon} - \int_{|x|=\epsilon^{-1}} \right) dx \lambda(x^{-1}Z)^{-1} f(x) x^{-1}, \quad (5)$$

where ϵ is small enough that the interior of the contour includes all the weights of Z . To verify this formula, notice that it suffices to assume that Z has distinct weights, and that Cauchy's residue formula leads to

$$\text{Res}_{x \in Z} \lambda(x^{-1}Z)^{-1} f(x) x^{-1}, \quad \text{Res}_{x \in Z} = \sum_{x \in Z} \text{Res}_x,$$

which is precisely formula (1) for this example. On the other hand, we may add the residues in the complementary region, and notice that

$$\begin{aligned} \chi^0(f(\mathcal{U})) &= -\text{Res}_{x=\infty} \lambda(x^{-1}Z)^{-1} f(x) x^{-1}, \\ \chi^d(f(\mathcal{U})) &= -\text{Res}_{x=0} \lambda(x^{-1}Z)^{-1} f(x) x^{-1}, \end{aligned} \quad (6)$$

where we have extended $\chi^i(\mathcal{E}) = \text{ch } H^i(\mathcal{E})$ by linearity to a virtual bundle given by $f(\mathcal{U})$, for any virtual character $f \in \mathbb{C}[x^{\pm 1}]$.

There is a standard way to reduce the general case to $n = 1$, which is a simple instance of a theorem of Shaun Martin [33] for general

symplectic quotients, or the Jeffrey-Kirwan residue formula [23]. In this case, it says

$$\chi_X(f(\mathcal{U})) = \frac{1}{n!} \operatorname{Res}_{x_1 \in Z} \cdots \operatorname{Res}_{x_n \in Z} \lambda(\bar{x}Z)^{-1} \Delta(x) f(x) \det(x)^{-1}, \quad (7)$$

where

$$x = \{x_1, \dots, x_n\}, \quad \bar{x} = \{x_1^{-1}, \dots, x_n^{-1}\},$$

$$\Delta_x = \lambda \left(\sum_{i \neq j} x_i x_j^{-1} \right), \quad \det(x) = x_1 \cdots x_n,$$

and we have identified f with its character restricted to the maximal torus $\{\operatorname{diag}(x_1, \dots, x_n)\} \subset GL_n$

3 The projection formula

Suppose $T = \mathbb{C}^*$ is a one-dimensional complex torus with parameter z acting on an infinite-dimensional complex vector space Z . Suppose furthermore that there exists M, W such that

$$Z = WM^{-1} \in \mathbb{Z}_{\geq 0}((z)), \quad W \in \mathbb{Z}[z^{\pm 1}], \quad M \in \mathbb{Z}[z].$$

Without loss of generality, we may normalize so that M is monic, making the two polynomials unique. One source of such representations is to choose a polynomial algebra R with a torus action encoded by a $\mathbb{Z}_{\geq 0}$ grading, and to let Z be the total space of an equivariant module. Then the character of R is given by M^{-1} for some monic polynomial, and W is determined by the grading of the resolution.

Let $X = G(n, Z)$ be the ind-Grassmannian of codimension n subspaces of Z , taken as a limit of subspaces

$$X = \bigcup_k X_k, \quad X_k = G(n, Z_k),$$

where Z_k is the direct sum of the subspaces with torus weight $a \leq k$. We set

$$\tilde{K}_T(X) = \varprojlim \tilde{K}_T(X_k), \quad \tilde{K}_T(X_k) = K_T(X) \otimes \mathbb{C}[[z]].$$

The above inverse system is determined by the pullback maps i_{ab}^* where

$$i_{ab} : X_a \rightarrow X_b, \quad V \mapsto \pi_{ba}^{-1}(V), \quad a \leq b,$$

and $\pi_{ab} : Z_b \rightarrow Z_a$ is the projection map.

The bundle $\mathcal{E} = \rho(\mathcal{U})$ defines an element of $\tilde{K}_T(X)$, as does the tangent bundle T^*X , with

$$i_a^*(T^*X) = \text{Hom}(\mathcal{U}, \mathcal{V} \oplus (Z/Z_a)).$$

Given a class $\gamma \in K_T(X)$, we define

$$\chi_X(\gamma) = \lim_{k \rightarrow \infty} \chi_{X_k}(i_k^* \gamma) \in \mathbb{C}[[z^{\pm 1}]],$$

whenever the sequence is convergent.

Now choose another infinite-dimensional representation B such that $A = Z - B$ is a Laurent polynomial, and define a virtual bundle

$$\mathcal{Y}^* = B\mathcal{U}^* - (1 - M)\mathcal{U}\mathcal{U}^* \in \tilde{K}_T(X), \quad (8)$$

We will imagine that \mathcal{Y} is dual to some subvariety $Y \subset X$, and make the definition

$$\chi_Y(\mathcal{E}_Y) = \sum_{C \subset X^T} \chi_C \left(\mathcal{E}_C \lambda(\mathcal{Y}_C^*) \lambda(\mathcal{N}_{X/C}^*)^{-1} \right) \quad (9)$$

in analogy with (1), (2). In reality, the variety may not actually exist, but we will see in the examples that \mathcal{Y}^* can be conjured so that (9) agrees with localization on some desirable variety.

If U is a torus representation with finite-dimensional weight spaces, let $\dim_a(U)$ denote the dimension of the subspace of U corresponding to the weight a . If $W = U - V$ is a virtual torus representation, let us define

$$\dim_a(W) = \dim_a(U) - \dim_a(V), \quad \dim(W) = \sum_a \dim_a(W),$$

provided $\dim_a(W) = 0$ for all but finitely many a .

Theorem 1. *Given the above notations, suppose the following conditions are satisfied:*

1. *We have*

$$\mathcal{E} \subset \det(\mathcal{U})^{1-l} \mathcal{U} \otimes \cdots \otimes \mathcal{U}, \quad l = \dim(Z - B).$$

2. For any torus weights a, b with $\dim_a(Z) > 0$, $\dim_a(M) < 0$, we have

$$\dim_{a+b}(B) \geq \dim_a(Z) - \dim_b(M) - 1.$$

3. For each fixed component $C \subset G^T$, $\lambda(\mathcal{Y}_C^*)$ is well-defined in the sense of (3), and vanishes for all but finitely many C .

Then we have the projection formula,

$$\chi_Y(\mathcal{E}_Y) = \sum_{j \geq 0} (-1)^j \chi_X(\mathcal{E} \lambda^j(\mathcal{Y}^*)). \quad (10)$$

Before beginning the proof, we need some technical lemmas. The key is lemma 3, which establishes that the characters of the higher cohomology groups have very large degree in z , and so vanish as k tends to infinity.

Given a parameter w , a set of variables x , and fixed value of $|z| < 1$, let us set

$$\mathcal{X}_k(x) = \mathcal{A}_k(x) \mathcal{B}(x), \quad \mathcal{A}_k(x) = \det(x)^{-l} \lambda(wB) \lambda(Z_k \bar{x})^{-1},$$

$$\mathcal{B}(x) = \frac{1}{n!} \Delta_x \lambda(w(M-1)x \bar{x}),$$

and let $f \in \Lambda$ be a symmetric polynomial so that $\mathcal{E} = \det(\mathcal{U})^{1-l} f(\mathcal{U})$. Here w is treated as a plethystic variable for the purposes of definition (4), rather than a constant. Then by equation (7), we have

$$\sum_j (-1)^j w^j \chi_{X_i}(\mathcal{E} \lambda^j(\mathcal{Y}^*)) = \text{Res}_{x_1 \in Z_k} \cdots \text{Res}_{x_n \in Z_k} f(x) \mathcal{X}_k(x), \quad (11)$$

for small enough values of w . We are interested in its analytic continuation to $w = 1$.

Lemma 1. *Fix complex values for w, z with $|z| < 1$, and suppose x consists of just one variable. Under the conditions of the theorem, $\mathcal{A}_k(x)$ is holomorphic in the plane except for simple singularities at the weights $z^a \in Z_k$ of order $\dim_a(Z)$, and an essential singularity at the origin. Their Laurent series satisfy*

$$\mathcal{A}_k(x) = \sum_j b_j(z) x^j, \quad \nu(b_j(z)) \geq -jk + c,$$

$$\mathcal{A}_k(x) = \sum_j b_j(z)(x - z^a)^j, \quad \nu(b_j(z)) \geq -aj + c,$$

ν is the valuation on $\mathbb{C}((z))$ given by $\nu(z^i) = i$, $l = \dim(A)$, and c is independent of a, k .

Proof. Since B is an honest representation, $\lambda(wBx^{-1})$ is holomorphic except at the origin, so the only singularities are the ones considered in the lemma.

Let us start with the residue at zero. Let v_i denote the weights of Z , sorted so that $\nu(v_i) \leq \nu(v_{i+1})$. We repeat entries to count for weight multiplicities so that

$$Z_k = v_1 + \cdots + v_m, \quad m = \dim(Z_k).$$

Let u_i denote the corresponding weights of B .

We have

$$\begin{aligned} x^l \mathcal{A}_k(x) &= \lambda(wBx^{-1}) \lambda(vx^{-1})^{-1} = \\ &= \det(v)^{-1} x^m \lambda(wux^{-1}) \lambda(\bar{v}x)^{-1} = \\ &= v_1^{-1} \cdots v_m^{-1} \sum_{i,j} (-1)^i x^{m-i+j} e_i(wu) h_j(\bar{v}) = \\ &= v_1^{-1} \cdots v_m^{-1} \sum_{i,j} (-1)^i x^i e_{m+j-i}(wu) h_j(\bar{v}) = \\ &= \sum_i x^i b_i(z), \quad \nu(b_i(z)) \geq \min_{j \geq 0} b_{ij}, \end{aligned}$$

where

$$\begin{aligned} b_{ij} &= (c_1 + \cdots + c_{m+j-i}) - (d_1 + \cdots + d_m) - jk, \\ c_i &= \nu(u_i), \quad d_i = \nu(v_i). \end{aligned}$$

Since $A = Z - B$ is a Laurent polynomial, we have that $u_i = v_{i+l}$ i large enough. By the ambiguity in the constant c , we may assume that this holds for all i . In that case, the minimum value of b_{ij} always occurs at $j = \max(0, i - l)$. In either case we see that

$$\nu(b_i(z)) \geq c + (l - j)k.$$

The first equality holds by reinserting the factor of x^l .

As for the second expansion, notice that

$$\lambda(z^b x^{-1}) = \sum_j b_j(z)(x - z^a)^j, \quad b_j(z) = 1 - z^{b-a-aj}(x - z^a)^j.$$

Then

$$\nu(b_j(z)) \geq \begin{cases} -aj & a \leq b \\ b - a - aj & a > b \end{cases}$$

Since $\mathcal{A}_k(x)$ is a product of powers of such terms, its coefficients must also be of the form $-aj + c$, where c is the sum of the terms of $\nu(b_j(z))$ in of its factors that are constant in j . It remains to show that the constant is bounded below for all a , which follows from the assumption that $A = Z - B$ is a Laurent polynomial. \square

Lemma 2. *For any $i \neq j$, and w not a power of z , we have*

$$\text{Res}_{x_i=z^a} \text{Res}_{x_j=wz^b x_i} f(x) \mathcal{X}(x) = 0.$$

Proof. We may ignore $f(x)$. If $c = -\dim_b(M)$, then

$$\text{Res}_{x_j=wz^b x_i} \mathcal{X}_k(x) = \frac{1}{(c-1)!} \partial_{x_j}^{c-1} \big|_{x_j=wz^b x_i} (x_j - wz^b x_i)^c \mathcal{X}_k(x).$$

We only need to show that

$$\mathcal{A}_k(x_i) \partial_{x_j}^{c-1} \big|_{x_j=wz^b x_i} \mathcal{A}_k(x_j)$$

is holomorphic at $x_i = z^a$ for any k . But this is precisely condition 2. \square

Lemma 3. *Let $b_k(z, w) = a_k(z, w) - a_k^0(z, w)$, where*

$$a_k^\epsilon(z, w) = \sum_i (-1)^i w^i \chi_{X_k}^\epsilon (f(\mathcal{U}) \lambda^i(\mathcal{Y}^*)), \quad (12)$$

and ϵ denotes the empty symbol or 0. Then $b_k(z, w)$ is the expansion of an analytic function in the range

$$0 < |z| < 1, \quad w \in \mathcal{D},$$

$$\mathcal{D} = \{|w| < |z|^{-1} : w \neq z^a, a \leq k\},$$

and for each $w \in \mathcal{D}$ we have

$$\lim_{k \rightarrow \infty} \nu(b_k(z, w)) = \infty.$$

Proof. We will prove this using the contour integral descriptions from section 2. By (6) and (11), we may rewrite $b_k(z, w)$ as

$$b_k(z, w) = \oint_{|x_1|=r} dx_1 \cdots \oint_{|x_n|=r} dx_n f(x) \mathcal{X}_k(x) - \text{Res}_{x_1 \in Z_k} \cdots \text{Res}_{x_n \in Z_k} f(x) \mathcal{X}_k(x). \quad (13)$$

for large enough r .

By Cauchy's residue formula, we have

$$\text{Res}_{x_i \in Z_k} \mathcal{X}_k(x) = \left(\oint_{|x_i|=r} dx_i - \oint_{|x_i|=\epsilon} dx_i \right) f(x) \mathcal{X}_k(x), \quad (14)$$

as long as $\epsilon < |z^k|$, and w is small enough that the contour includes all of the poles of $\mathcal{A}_k(x)$, but none of the poles of $\mathcal{B}(x)$. Using this relation, we may express $b_k(z, w)$ as a linear combination of expressions of the form

$$c_k(z, w) = \oint_{|x_1|=\epsilon} dx_1 \cdots \oint_{|x_c|=\epsilon} dx_c \text{Res}_{y_1=z^{a_1}} \cdots \text{Res}_{y_d=z^{a_d}} f(x \cup y) \mathcal{X}(x \cup y), \quad c + d = n, \quad a_i \neq a_j, \quad (15)$$

and $c \geq 1$.

Consider the analytic continuation of $c_k(z, w)$ to \mathcal{D} in the w variable. As w varies, the singularities of $\mathcal{B}(x \cup y)$ at $y_j = wz^b x_i$ may cross the contour $|y_j| = \epsilon$. However, it follows from lemma 2 that these additional residues have no contribution, which means that expression (15) describes $c_k(z, w)$ for all $w \in \mathcal{D}$. It suffices to show that the valuation of $c_k(z, w)$ tends to infinity as k tends to infinity uniformly in a_i for all w in some subregion $\mathcal{R} \subset \mathcal{D}$; the valuation cannot drop below this value under analytic continuation in w , because each coefficient in the Laurent series about $z = 0$ is an analytic function of w .

We define \mathcal{R} as follows: pick a grading on monomials by

$$\deg(z) = 1, \quad \deg(x_i) = k, \quad \deg(y_i) = a_i.$$

Let \mathcal{R} be a region such that

$$\deg(v) \leq 0 \Rightarrow |wv| < 1, \quad \deg(v) > 0 \Rightarrow |wv| > 1,$$

whenever v is a monomial of the form

$$z^b y_i y_j^{-1}, \quad z^b x_i^{-1} y_j, \quad z^b x_i y_j^{-1}, \quad \dim_b(M) < 0,$$

and x_i, y_j are any points on their respective contours. Such a region exists provided ϵ near enough to $|z^k|$, and the contours representing the residues at $y_j = z^{a_j}$ are confined to a small enough neighborhood of z^{a_j} .

We may replace any factor in $\mathcal{X}(x \cup y)$ by its Laurent series in x_i, y_j about any point, as long as the contours lie in its domain of convergence. If $w \in \mathcal{R}$, the expansions may be taken as follows:

1. Expand $\mathcal{A}(x_i)$ about $x_i = 0$.
2. Expand $\mathcal{A}(y_i)$ about $y_i = z^{a_i}$.
3. Expand each factor in $\mathcal{B}(x \cup y)$ about either zero or infinity, depending on which expansion is valid (exactly one can be).

By the definition of \mathcal{R} , the degree of each term in the expansions of type 3 is nonnegative. By lemma 1, the degree of each term in the expansions of types 1 and 2 is bounded below by some overall constant independent of a_i . The lemma follows because there is at least one term of the form $\mathcal{A}(x_i)$, and the residue about $x_i = 0$ raises the degree by k .

□

We may now prove the theorem.

Proof. Let

$$a(z) = \lim_{k \rightarrow \infty} \lim_{w \rightarrow 1} a_k(z, w) \in \mathbb{C}[[z^{\pm 1}]],$$

which exists by condition 3. The limit over k is applied termwise to the Laurent series of a meromorphic function of z , and the limit over w refers to analytic continuation. Then $a(z)$ agrees with the left hand side of (10).

Similarly, let

$$a^0(z) = \lim_{k \rightarrow \infty} a_k^0(z, 1).$$

Using the definition of \mathcal{Y}^* , and that the weights of Z are bounded below, we can see that

$$\lim_{j \rightarrow \infty} \nu(\chi_{X_k}^0(\mathcal{E}\lambda^j(\mathcal{Y}^*))) = \infty$$

uniformly in k . It follows that $a^0(z)$ agrees with the right hand side of (10). By lemma 3, we have

$$a(z) = \lim_{w \rightarrow 1} \lim_{k \rightarrow \infty} a_k(z, w) = a^0(z) \in \mathbb{C}[[z^{\pm 1}]],$$

proving the theorem.

□

4 Examples

4.1 The Hilbert scheme of points in the plane

Let $Y = \text{Hilb}_n \mathbb{C}^2$, the Hilbert scheme of n points in the plane. There is a standard torus action on Y induced by pullback of ideals from the action on the plane

$$(z_1, z_2) \cdot (x, y) = (z_1^{-1}x, z_2^{-1}y) \quad (16)$$

The fixed points of Y are the monomial ideals indexed by Young diagrams

$$I_\mu = (x^{\mu_1}, x^{\mu_2-1}y, \dots, y^{\ell(\mu)}) \subset R = \mathbb{C}[x, y].$$

The character of the cotangent space to this point is a polynomial in z_i with nonnegative integer coefficients summing to $\dim(Y) = 2n$. By deformation theory and a standard Čech cohomology argument, it is given by

$$T_\mu Y = \chi(R, R) - \chi(I_\mu, I_\mu), \quad (17)$$

where χ is the Euler characteristic

$$\chi(F, G) = \sum_i \text{ch Ext}_R^i(F, G).$$

There is an interesting formula for this polynomial in terms of the arm and leg lengths of boxes in μ , which we will not need. See [10, 36] for a reference on this calculation.

Now let Z be the total space of R , so that

$$Z = M^{-1}, \quad M = (1 - z_1)(1 - z_2), \quad B = Z - 1.$$

Let $X = G(n, Z)$, and

$$V_\mu = H^0(I_\mu) \subset Z, \quad U_\mu = Z/V_\mu = U_\mu = \sum_{(i,j) \in \mu} z_1^j z_2^i,$$

where (i, j) are the coordinates of a box in μ . The following lemma shows that this data represents the fundamental class of the Hilbert scheme $Y \subset X$ in the sense described in the introduction.

Lemma 4. *Let V be an invariant subspace $V \subset Z$, with complement $U = Z - V$ of dimension n . Then*

- a. We have $\lambda(MT_V^*X)$ vanishes unless V is the total space of an invariant ideal represented by a Young diagram μ .
- b. If $V = V_\mu$, then

$$MT_\mu^*G(n, Z) + z_1 z_2 U_\mu = T_\mu^* \text{Hilb}_n \mathbb{C}^2.$$

Proof. For part a, it suffices to show that the constant term of

$$(1 - M)T_V^*X = (1 - M)\overline{U}V$$

is positive unless $V = V_\mu$, in which case it is zero. Consider the graph whose vertices are $\mathbb{Z}^2 \subset \mathbb{R}^2$, and whose edge set E consists of horizontal and vertical neighbors. Color each box with lower-left corner (i, j) white if $z_1^i z_2^j$ is a weight of V , and black otherwise. Define subsets by

$$X_0 = \{v \in \mathbb{Z}^2 : v(\nearrow) \text{ is black, } v(\swarrow) \text{ is white}\}$$

$$X_1 = \{e \in E : e(\nearrow) \text{ is black, } e(\swarrow) \text{ is white}\}$$

Here $v(\nearrow)$ is the upper-right neighboring box to v , $e(\nearrow)$ is the upper or right neighboring box to the edge e depending on whether e is horizontal or vertical, and similarly for the southwest arrow.

Expanding M , we see that the constant term is

$$[z_1^0 z_2^0](1 - M)T_V^*X = x_1 - x_0, \quad x_i = |X_i|.$$

Now, notice that every vertex in X_0 is the endpoint of exactly two edges in X_1 , but each edge in X_1 always has at most two endpoints in X_0 , proving that $x_1 - x_0 \geq 0$. If U does not come from a Young diagram, then the set X_1 is not empty, and there must be some edge in X_1 whose endpoints are not both in X_0 , leading to strict inequality.

The equation relating MT_μ^*X to the cotangent bundle of the Hilbert scheme may be deduced easily from (17), and the rule that

$$\chi(I_\mu, I_\nu) = z_1^{-1} z_2^{-1} M \overline{V}_\mu V_\nu,$$

proving part b. □

Now restrict to a one-dimensional torus $z_i = z^{a_i}$, where the a_i are large enough that the fixed points of Y are isolated. Lemma 4 combined with localization on C with respect to the two-dimensional

torus proves that condition 3 of the theorem is satisfied. Condition 2 follows from the fact that Z has the same character as R , and B has the same character as the total space of its maximal ideal \mathfrak{m} ; it is just the statement that xR and yR are contained in \mathfrak{m} .

By theorem 1, we have the following formula for the Euler characteristic of a polynomial functor applied to the tautological n -dimensional bundle on the Hilbert scheme

$$\chi_Y(f(\mathcal{U}_Y)) = \sum_j \chi_X(\lambda(z_1 z_2 \mathcal{U})^{-1} f(\mathcal{U}) \lambda^j(\mathcal{Y}^*)), \quad (18)$$

where

$$\begin{aligned} \mathcal{Y}^* &= (z_1 + z_2 - z_1 z_2) T^* X, \\ \lambda(x\mathcal{U})^{-1} &= [1 + x\mathcal{U} + x^2 \text{Sym}^2(\mathcal{U}) + \cdots], \end{aligned}$$

and \mathcal{U}_Y is the tautological rank n bundle on Y , which is pulled back from X . Since both sides are rational functions of z_i , they are determined by their values on the restricted torus. We may therefore drop the assumption that $z = z^{a_i}$, and have an equality of functions of two distinct torus variables z_i . The left hand side of (18) represents localization on the Hilbert scheme with respect to the general two-dimensional torus, whereas the right hand side is a power series with integer coefficients in $\mathbb{Z}[[z_1, z_2]]$.

4.2 The moduli space of sheaves

The same applies when Y is the more general moduli space of framed torsion-free sheaves on \mathbb{P}^2 denoted $\mathcal{M}_{r,n}$, which can be found in [21, 36]. This space is not naturally embedded in the Grassmannian the way the Hilbert scheme is, but we still obtain the following generalization of (18):

$$\chi_{\mathcal{M}_{r,n}}(f(\mathcal{U})) = \sum_j \chi_X(f(\mathcal{U}) \lambda(z_1 z_2 W \mathcal{U})^{-1} \lambda^j(\mathcal{Y}^*)), \quad (19)$$

$$M = (1 - z_1)(1 - z_2), \quad Z = W M^{-1},$$

$$W = w_1 + \cdots + w_r, \quad B = Z - W,$$

where w_i are the torus variables coming from an r -dimensional action on the framing, as in [40].

4.3 The Hilbert scheme of a singular curve

Let C denote the singular curve $y^2 = x^3$, and consider the action

$$T = \mathbb{C}^* \curvearrowright C, \quad z \cdot (x, y) = (z^{-2}x, z^{-3}y).$$

Let Y denote the Hilbert scheme of n points in this curve, whose points correspond to ideals in

$$R = \mathbb{C}[x, y]/(y^2 - x^3) \cong \mathbb{C}[u^2, u^3].$$

with $\dim_{\mathbb{C}} R/I = n$. The torus fixed points of Y are those of the form

$$I_S = \bigoplus_{i \in S} \mathbb{C} \cdot u^i \subset R,$$

for S a sub-semigroup of $\{0, 2, 3, 4, \dots\}$.

This moduli space is an lci subvariety of $\text{Hilb}_n \mathbb{C}^2$, leading to the following definition for the virtual cotangent bundle: Let \mathcal{L} be the trivial bundle on \mathbb{C}^2 with character z^{-6} , so that C is the intersection of the zero section with equivariant section $y^2 - x^3$. Any line bundle on a surface S induces a rank n bundle $\mathcal{L}^{[n]}$ on the Hilbert scheme by pulling back to the canonical subvariety

$$Z \subset \text{Hilb}_n S \times S,$$

and pushing forward over the degree n map to the Hilbert scheme. In this case $\mathcal{L}^{[n]}$ is the tautological bundle \mathcal{U} twisted by the torus character z^{-6} . We can verify that $\text{Hilb}_n C \subset \text{Hilb}_n \mathbb{C}^2$ is the zero set of the induced section of $\mathcal{L}^{[n]}$, by considering the closure of the portion of this variety in the complement of the big diagonal. Its fundamental class is represented by $\lambda(z^6 \mathcal{U}^*)$. For a reference, see [48].

The character of the virtual cotangent bundle to an equivariant ideal I_S is therefore given by

$$\begin{aligned} T_S^* \text{Hilb}_n C &= T_S^* \text{Hilb}_n \mathbb{C}^2 - z^6 \mathcal{U}_S^* = \\ M \overline{U}_S (M^{-1} - U_S) + z^5 U_S - z^6 \overline{U}_S &= \\ MT_S^* X + z^5 U_S, \end{aligned} \tag{20}$$

where

$$\begin{aligned} M &= (1 - z^2)(1 - z^3), \quad A = 1 - z^6, \\ Z &= AM^{-1} = \mathbb{C}[u^2, u^3], \quad B = Z - A, \end{aligned}$$

U_S is the character of R/I_S , and $X = G(n, Z)$. This is a virtual bundle, which means at singular points we may not obtain a polynomial with nonnegative coefficients, but the signed dimension will always equal the expected dimension of n .

We may check that this choice of Z, M satisfy the conditions of theorem 1, and that $\lambda(y^*)$ vanishes on a codimension n invariant subspace $V \subset Z$ unless $V = I_S$ for some semigroup S . We obtain the following:

$$\chi_{\text{Hilb}_n C}(f(\mathcal{U})) = \sum_j \chi_X \left(f(\mathcal{U}) \lambda(z^5 \mathcal{U})^{-1} \lambda^j(y^*) \right), \quad (21)$$

where X is the Grassmannian of codimension n subspaces of Z , and $f \in \Lambda$ is a multiple of $e_n^{1-l} = e_n$.

4.4 The affine Grassmannian

Let $G = SL(2, \mathbb{C})$, and consider the affine Grassmannian

$$Y = LG_{\mathbb{C}}/L^+G_{\mathbb{C}},$$

where LG is the space of maps from the circle into G , and L^+G are those maps which extend to a holomorphic function in the disc of radius 1.

In [47], Segal noted that there should be a proof of the Weyl-Kač character formula using this variety, which is analogous of the well-known geometric proof of the Weyl character formula using K -theoretic localization combined with Borel-Weil-Bott, see [12]. He also pointed out that there was a gap in the reasoning due to the fact that Y is infinite-dimensional, and the explanation that the higher cohomology groups vanish. We would now like to demonstrate how theorem 1 can be used to circumvent these two difficulties, leading to a proof of the Weyl-Kač character formula, in case of the basic representation of the loop group of $SL(2, \mathbb{C})$. This approach obviously generalizes, and it would be interesting to see if one can recover the complete formula this way. This topic, and generalizations to the related flag varieties have studied by several authors, including [26, 49, 50].

Let us explain the moral calculation, ignoring technicalities: there is an action of a two-dimensional torus by

$$(g \cdot f)(x) = \text{Ad} \left(\begin{pmatrix} z^{-1} & \\ & z \end{pmatrix} \right) \cdot (f(qx)), \quad g = (q, z) \in (\mathbb{C}^*)^2.$$

The fixed points are precisely the orbits of the Weyl group elements $w \in LG_{\mathbb{C}}$ [47]. Ignoring the infinite-dimensionality of Y , we can write down the character of the cotangent bundle to this space at a fixed point w as follows. The cotangent bundle at the image of the identity in Y is given by

$$T_1^*Y = (L\mathfrak{g}_{\mathbb{C}}/L^+\mathfrak{g}_{\mathbb{C}})^* = \frac{q}{1-q}(z^2 + 1 + z^{-2}).$$

The character at a general fixed point can be extracted from by applying elements of the affine Weyl group.

Let $\mathcal{H} = L\mathbb{C}^2$, the Hilbert space of maps to \mathbb{C}^2 . Then $LG_{\mathbb{C}}$ acts in the obvious way on this space, and $L^+G_{\mathbb{C}}$ is precisely the subgroup that preserves the subspace $V \subset \mathcal{H}$ of all maps which are holomorphic at the origin. The action on \mathcal{H} induces an imbedding $Y \subset X$, where X is the Sato Grassmannian of half-infinite dimensional subspaces of \mathcal{H} , by taking the orbit space of V . The character of the cotangent bundle at V is given by

$$\text{ch } T_V^*X = \frac{q}{(1-q)^2}(z^2 + 2 + z^{-2}),$$

which is the character of one quadrant of $\text{End}(\mathcal{H})$. We therefore have

$$\text{ch } T_1^*Y = (1-q)T_V^*X - \frac{q}{1-q}.$$

Now let us explain how the Kaç character formula follows from theorem 1, in this special case. For each n , let

$$M = 1 - q, \quad A = q^{-n}(z + z^{-1}), \quad Z = AM^{-1},$$

$$B = Z - A, \quad X = G(2n, Z),$$

and notice $Z \subset \mathcal{H}$, and includes the whole space as n becomes large. We may check that $\lambda(\mathcal{Y}^*)$ vanishes at all points of X except those whose complementary subspace has character

$$U_k = \sum_{-n \leq i \leq k-1} zq^i + \sum_{-n \leq i \leq -k-1} z^{-1}q^i.$$

We can see by direct calculation that

$$a_k = \lim_{n \rightarrow \infty} \lambda(MT_k^*X)^{-1} =$$

$$(q; q)_\infty^{-1} \theta(z^2, q)^{-1} \left(z^{4k} q^{2k^2+k} - z^{4k-2} q^{2k^2-k} \right), \quad (22)$$

where

$$(x; q)_\infty = \prod_{i \geq 0} (1 - xq^i), \quad \theta(x; q) = (q; q)_\infty (xq; q)_\infty (x^{-1}; q)_\infty.$$

Just as in section 4.1, the fact that the torus is two-dimensional is not an issue, because two rational functions that agree at $q = z^a$ for infinitely many values of a must be equal. The projection formula therefore applies, and it says that

$$\sum_k a_k = \sum_j (-1)^j q^j \lim_{n \rightarrow \infty} \chi_X (\lambda^j(T^*X)) = (q; q)_\infty^{-1}. \quad (23)$$

The second equation follows from

$$\chi_{\text{Gr}(k, n)} (\lambda^j(T^*)) = (-1)^j p(j),$$

where $\text{Gr}(k, n)$ may be equivariant with respect to any torus action on \mathbb{C}^n , $p(j)$ is the number of partitions of j , and n , $n - k$ are sufficiently large. Combining (22) and (23), we may read off the Jacobi triple product, which is the simplest instance of the Kač character formula for the A_1 root system.

To obtain the more general character formula, we would need to consider arbitrary representations applied to the two-dimensional virtual bundle $(1 - q)\mathcal{V}$.

References

- [1] L.F. Alday, D. Gaiotto, Y. Tachikawa, *Liouville correlation functions from four dimensional gauge theories*, Lett. Math. Phys. 91 (2010), 167-197.
- [2] M. Atiyah, R. Bott, *The Moment map and equivariant cohomology*, Topology 23 (1984), no. 1, 1 - 28.
- [3] M. Atiyah, V.G. Drinfel'd, N.J. Hitchin, and Y.I. Manin *Construction of Instantons*, Phys. Lett. A65 (1978) 185-187
- [4] V. Baranovsky, *Moduli of sheaves on surfaces and action of the oscillator algebra*, J. Differential Geom. 55 (2000), no. 2, 193 - 227.

- [5] S. Bloch, A. Okounkov, *The character of the infinite wedge representation*, Adv. Math. 149 (2000), no. 1, 1 - 60.
- [6] T. Bridgeland, A. King, M. Reid, *The McKay correspondence as an equivalence of derived categories*, J. Amer. Math. Soc. **14** (2001), no. 3, 535–554.
- [7] E. Carlsson, *Vertex Operators and Moduli Spaces of Sheaves*, PhD Thesis, Princeton University,
- [8] E. Carlsson, *Vertex operators, Grassmannians, and Hilbert schemes* Comm. Math. Phys. 300, no. 3, (2010), 599-613.
- [9] E. Carlsson *Vertex operators and quasi-modularity of Chern numbers on the Hilbert scheme* (2011), Advances in Mathematics, to appear.
- [10] E. Carlsson, A. Okounkov,
Exts and vertex operators, [arXiv:0801.2565v1](#)
- [11] E. Carlsson, N. Nekrasov, and A. Okounkov, *Exts and vertex operators 2*, [arxiv preprint](#).
- [12] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Birkhuser Boston, Inc., Boston, MA, 1997.
- [13] G. Ellingsrud, L. Göttsche, M. Lehn, *On the cobordism class of the Hilbert scheme of a surface* Journal of Algebraic Geometry, **10** (2001), 81 - 100.
- [14] E. Frenkel, D. Ben-Zvi, *Vertex algebras and algebraic curves*, Mathematical Surveys and Monographs, vol. 88. AMS 2001.
- [15] L. Göttsche, *Hilbert schemes of points on surfaces*, ICM Proceedings, Vol. II (Beijing, 2002), 483–494.
- [16] L. Göttsche, *The Betti numbers of the Hilbert scheme of points on a smooth projective surface*, Math. Ann. 286 (1990), no. 1-3, 193 - 207.
- [17] I. Grojnowski, *Instantons and affine algebras I: the Hilbert scheme and vertex operators*, Math. Res. Lett. **3** (1996), 275–291.
- [18] V. Guilleman and S. Sternberg *Supersymmetry and equivariant de Rham theory*, Springer-Verlag Berlin Heidelberg, 1999
- [19] M. Haiman, *Hilbert schemes, polygraphs and the Macdonald positivity conjecture*, J. Amer. Math. Soc. 14 (2001), no. 4, 941-1006, [arXiv:math.AG/0010246](#).

- [20] M. Haiman, *Combinatorics, symmetric functions, and Hilbert schemes*, Current developments in mathematics, 2002, 39 - 111, Int. Press, Somerville, MA, 2003.
- [21] D. Huybrechts, M. Lehn, *The geometry of moduli spaces of sheaves*, Aspects of Mathematics, E31. Friedr. Vieweg & Sohn, Braunschweig, 1997.
- [22] A. Iqbal, C. Kozcaz, K. Shabbir, *Refined Topological Vertex, Cylindric Partitions and the $U(1)$ Adjoint Theory*, <http://arxiv.org/abs/0803.2260>.
- [23] L. Jeffrey, F. Kirwan, *Localization for nonabelian group actions* Topology Volume 34, Issue 2, April 1995, Pages 291-327
- [24] V. Kaç, *Infinite dimensional Lie algebras, third edition*, Cambridge University Press, 1990.
- [25] M. Kaneko and D. Zagier, *A generalized Jacobi theta function and quasimodular forms*, The moduli space of curves, Progress in Mathematics, **129**, Birkhäuser, 1995.
- [26] S. Kumar, *Demazure character formula in arbitrary Kac-Moody setting*, Inventiones mathematicae, 1987, Volume 89, Issue 2, pp 395-42.
- [27] M. Lehn, *Geometry of Hilbert schemes*, CRM Proceedings and Lecture Notes, Volume 38, 2004, 1 - 30.
- [28] M. Lehn, *Chern classes of tautological bundles on Hilbert schemes of points on surfaces*, Invent. Math. 136 (1999), no. 1, 157 - 207.
- [29] W. Li, Z. Qin, W. Wang, *Vertex algebras and the cohomology ring structure of Hilbert schemes of points on surfaces*, Math. Ann. **324** (2002), 105 - 133.
- [30] A. Licata, *Framed torsion-free sheaves on CP^2 , Hilbert schemes, and representations of infinite dimensional Lie algebras* Adv. Math, vol. 226, no. 2, pp. 1057-1095, 2011.
- [31] W. Li, Z. Qin, W. Wang, *The cohomology rings of Hilbert schemes via Jack polynomials*, CRM Proceedings and Lecture Notes, vol. **38** (2004), 249-258.
- [32] I. Macdonald, *Symmetric functions and Hall polynomials*, The Clarendon Press, Oxford University Press, New York, 1995.
- [33] S. Martin, *Cohomology rings of symplectic quotients*, arXiv:math/0001002 [math.SG].

- [34] D. Maulik and A. Okounkov, *Quantum Groups and Quantum Cohomology*, [arXiv:1211.1287](#) [[math.AG](#)]
- [35] H. Nakajima, *Heisenberg algebra and Hilbert schemes of points on projective surfaces*, *Ann. of Math.* (2) **145** (1997), no. 2, 379–388.
- [36] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, AMS, Providence, RI, 1999.
- [37] H. Nakajima, *Jack polynomials and Hilbert schemes of points on surfaces*, [arXiv:alg-geom/9610021](#)
- [38] H. Nakajima, *Instanton counting on blowup. I. 4-dimensional pure gauge theory*, *Invent. Math.* **162** (2005), no. 2, 313–355.
- [39] H. Nakajima, *Instanton counting on blowup. II: K-theoretic partition function*, [math.AG/0505553](#).
- [40] H. Nakajima, *Instantons and affine Lie algebras I*, *Nucl.Phys.Proc.Suppl.* **46**:154-161, 1996
- [41] A. Negut, *Laumon Spaces and the Calogero-Sutherland Integrable System*, *Inventiones Mathematicae*, Volume 178, Number 2 (Nov 2009), 299–331.
- [42] N. Nekrasov and A. Okounkov, *Seiberg-Witten Theory and Random Partitions*, In *The Unity of Mathematics* (ed. by P. Etingof, V. Retakh, I. M. Singer) *Progress in Mathematics*, Vol. 244, Birkhäuser, 2006, [hep-th/0306238](#).
- [43] A. Okounkov, *Random Partitions and Instanton Counting*, *International Congress of Mathematicians*. Vol. III, 687 - 711, Eur. Math. Soc., Zürich, 2006.
- [44] A. Okounkov and R. Pandharipande, *Quantum cohomology of the Hilbert scheme of points in the plane*, [arXiv:math/0411210](#).
- [45] A. Pressley, G. Segal, *Loop Groups*, Clarendon Press, Oxford, 1986.
- [46] S. Schiffmann, E. Vasserot, *The elliptic Hall algebra and the equivariant K-theory of the Hilbert scheme of \mathbb{A}^2* , [arXiv:0905.2555v2](#)
- [47] G. Segal, *Loop groups*, *Lecture Notes in Mathematics*, *Arbeitstagung Bonn 1984*, Subseries: Mathematisches Institut der Universität at und Max-Planck-Institut für Mathematik, Bonn - vol 5, chapter 8, 1984.

- [48] V. Shende, *A support theorem for Hilbert Schemes of planar curves*, [arXiv:1107.2355](#) [math.AG]
- [49] C. Teleman, *Borel-Weil-Bott theory for loop groups*, [arXiv:alg-geom/9707014](#)
- [50] C. Teleman, *Borel-Weil-Bott theory on the moduli stack of G -bundles over a curve*, *Inventiones mathematicae*, September 1998, Volume 134, Issue 1, pp 1-57.
- [51] E. Vasserot, *Sur l'anneau de cohomologie du schema de Hilbert de \mathbf{C}^2* , *C. R. Acad. Sci. Paris Sér. I Math.* **332** (2001), no. 1, 7 - 12.
- [52] E. Vasserot, *Cherednik algebras, W algebras and the equivariant cohomology of the moduli space of instantons on A^2* , [arXiv:1202.2756](#) [math.QA]
- [53] Y. Zhu, *Modular invariance of characters of vertex operator algebras*, *J. AMS*, **9** (1996), 237 - 302.